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A MATHEMATICAL MODEL OF AN ELASTIC-DAMPER SYSTEM BASED ON THE EXAMPLE OF A KELVIN-VOIGT BODY

Mechanical systems that combine structural elements with elastic and viscous properties and operate under dynamic and impulse loads require accurate mathematical models to predict their oscillatory behaviour. The purpose of this study was to develop a mathematical model of an elastic-damper system based on the Kelvin-Voigt body, aimed at describing viscoelastic material behaviour and analysing the influence of stiffness and damping parameters on the dynamic response of the system. The research applied analytical methods for solving differential equations, analysis of characteristic equations for both real and complex roots, as well as algebraic transformations for constructing general and particular solutions. As a result, an analytical solution was obtained for determining the dynamic response of the masses under impulse loading. The proposed model took into account variable stiffness, damping, and mass parameters, and distinguishes between regimes with real and complex eigenvalues. It was established that increasing the damping coefficient reduces amplitude and accelerates oscillation decay, while increasing mass extends the decay duration. The influence of pulse duration and repetition frequency on the oscillation profile was investigated. The simulation results confirmed the accuracy of the analytical expressions and enable the modelling of transient processes under various conditions. Compared to purely numerical approaches, the proposed method provides broader opportunities for system analysis and control. The developed model can be applied in the study of technical systems with arbitrary configurations of elastic-viscous element connections and varying numbers of components, making it possible to optimise the design of systems that utilise oscillations, vibrations, damping.

Keywords: *two-mass system, differential equations, characteristic equation, elasticity, damper.*

Introduction. The development of mathematical models of mechanical systems that take into account elastic-damping properties is a relevant direction in applied mechanics, as it enables more accurate reproduction of the dynamic behaviour of structures and technical objects under real operating conditions. Of particular importance are models capable of considering the impact of impulse loads, which are typical for many engineering tasks – from mechanical engineering to automatic control systems. The Kelvin-Voigt model, which combines elastic and viscous components, is used to analyse the motion of bodies where not only the moment of instantaneous impact is significant, but also the prolonged response of the system. In this context, an important scientific task is to study the response of mass-spring systems to the action of discrete impulse disturbances with variable duration and frequency, which allows the modelling of a wide range of dynamic regimes. This approach provides a new level of accuracy in the design of systems with damping and enables the adaptation of model parameters to the conditions of cyclic or intermittent loading.

Mechanical systems that have elements with elasticity and viscosity characteristics are extensively employed in engineering applications. The connection of such elements can be serial, parallel, or combined and makes it possible to implement the characteristics of a technical system that can perform the functions of regulation, oscillation, damping, etc. As demonstrated by Huilai *et al.*, the modelling of hydro-pneumatic suspensions using fractional calculus provides an accurate representation of damping and energy dissipation processes, which are essential for construction vehicles [1]. Lanets *et al.* developed an analytical model of a two-mass vibration system with eccentric-pendulum excitation, highlighting the dynamic advantages of such systems in resonance control [2]. The system of differential equations expressed in generalised coordinates can be transformed into an infinite set of decoupled equations, each representing damped oscillatory motion of a mechanical oscillator modelled according to the Kelvin-Voigt framework incorporating a fractional derivative.

To describe the rheological behaviour of soft materials with elastic-viscous characteristics, the method of fractional derivatives is used. Bonfanti *et al.* proposed a power-law-based fractional model that better reflects the non-linear stress–strain response of soft condensed matter, improving prediction accuracy in biomedical and materials science applications [3]. Modelling and research of vibrations and forced oscillations are carried out by the analogy of an elastic-damper element for various types of machines and equipment. The influence of dampers on the forced vibration of high-speed rotating blades is studied experimentally and numerically. Authors Wu *et al.* confirmed that under-platform dampers significantly reduce amplitude under resonance conditions and increase blade life in turbomachinery [4]. For the pneumatic suspension, the transfer functions were derived from the differential equations, and the characteristic was modelled computationally. R. Zhou *et al.* proposed a dynamic simulation approach that incorporates nonlinear stiffness characteristics of air springs and validated it against experimental data, confirming its suitability for vehicle dynamics simulation [5]. Also, previously known dependencies are used, and the system oscillations are numerically modelled – in particular, Leniowski & Wroński modelled the vibrations and oscillations of robot manipulator links [6]. The parallel connection of elastic and viscous elements has become widely used in vehicles. Satpute *et al.* conducted a numerical analysis of vibration transmission by a shock absorber [7]. They introduced a hybrid energy-harvesting system using a linear generator coupled with a motion amplification mechanism, which not only improves vibration attenuation but also generates electrical energy for onboard systems.

The analytical solution of the system of differential equations governing the behaviour of elastically and viscously coupled elements in parallel is often intractable, thereby requiring the application of approximate solution methods. This increases the error of the results obtained. The solution of the system of differential equations governing the motion of masses interconnected by a Kelvin-Voigt body represents a pertinent and timely research problem. Obtaining an analytical solution to such a system of differential equations for a two-mass configuration, wherein the external excitation is discrete rather than harmonic, enables the analytical optimisation of the system's parameters.

The aim of the research. The research purpose was to develop a mathematical model of an elastic-damping system based on the Kelvin-Voigt body and to analyse the influence of stiffness and damping parameters on the dynamic response of the system.

Materials and Methods. To model the dynamic response of the system to impulse excitation, a generalised mathematical model of a two-mass oscillating system was used, which accounted for the visco-elastic properties of the connecting elements according to the Kelvin-Voigt model. This approach made it possible to describe the behaviour of a mechanical system comprising two masses connected by elements possessing both elastic and damping properties. The model of the Kelvin-Voigt body, which is shown in Fig. 1, was considered.

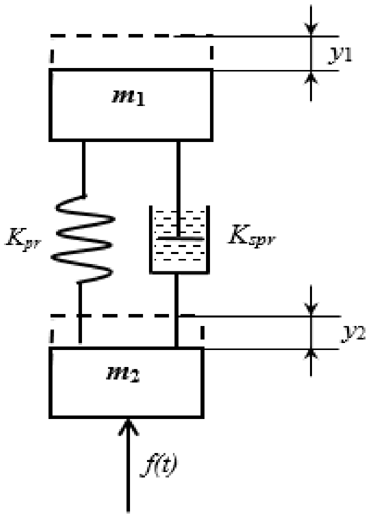

Fig. 1. A parallel connection of elastic and damping elements – Kelvin-Voigt body:
y_1, y_2 – amplitude of oscillation according to the mass m_1 and m_2; K_{spr} – the complex coefficient of resistance of the damping element; K_{pr} – the coefficient of elasticity of the elastic element; $f(t)$ – the characteristic of the oscillation of the applied force
Source: developed by the authors

Two masses m_1 and m_2 were connected by elastic K_{pr} and damping K_{spr} elements, which were placed parallel to each other. A disturbing force $f(t)$ acted on the lower mass m_2. Accordingly, the lower mass m_2 moved to a distance of y_2 and the upper mass m_1 moved to a distance of y_1. A system of differential equations characterising the action of forces on the Kelvin–Voigt body was written down:

$$\begin{cases} m_1 \cdot \frac{d^2y_1}{dt^2} + K_{spr} \cdot \left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) + K_{pr} \cdot (y_1 - y_2) = 0 \\ m_2 \cdot \frac{d^2y_2}{dt^2} - K_{spr} \cdot \left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) - K_{pr} \cdot (y_1 - y_2) = f(t) \end{cases} . \quad (1)$$

Similarly, the following notations were introduced: $K_1^2 = K_{pr}/m_1$, $K_2^2 = K_{pr}/m_2$ – the square of the frequency of free oscillations, respectively, of the elements by mass of m_1 and m_2, s^{-2}; $n_1 = K_{spr}/m_1$, $n_2 = K_{spr}/m_2$ – the coefficient of the oscillation dying, s^{-1}; $h_2 = 1/m_2$ – specific amplitude of force of constrained oscillations, m/s^2. Then, the set of equations (1**)** was rewritten as follows**:**

$$\begin{cases} \frac{d^2y_1}{dt^2} + n_1 \cdot \left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) + K_1^2 \cdot (y_1 - y_2) = 0 \\ \frac{d^2y_2}{dt^2} - n_2 \cdot \left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) - K_2^2 \cdot (y_1 - y_2) = h_2 \cdot f(t) \end{cases}. \quad (2)$$

Solving this system allowed the determining the nature of movements under the action of an external disturbing force. Although Euler's method could have been applied, the substitution method was used to simplify transformations and calculations. The following substitutions were made:

$$y_1 - y_2 = u, \dot{y}_1 - \dot{y}_2 = \dot{u}, \ddot{y}_1 - \ddot{y}_2 = \ddot{u}. \quad (3)$$

The second equation of system (2) was subtracted from the first element by element, and substitution (3) was applied. As a result, the following differential equation was obtained**:**

$$\ddot{u} + \dot{u} \cdot (n_1 + n_2) + u \cdot (K_1^2 + K_2^2) = -h_2 \cdot f(t). \quad (4)$$

The characteristic equation corresponding to differential equation (4) was obtained:

$$\lambda^2 + \lambda \cdot (n_1 + n_2) + (K_1^2 + K_2^2) = 0. \quad (5)$$

The solutions of the characteristic equation (5) were calculated as:

$$\lambda_{1,2} = -\frac{n_1 + n_2}{2} \pm \frac{\sqrt{(n_1 + n_2)^2 - 4 \cdot (K_1^2 + K_2^2)}}{2},$$

or

$$\lambda_{1,2} = -\frac{K_{spr}(m_2 + m_1)}{2 \cdot m_1 \cdot m_2} \pm \frac{\sqrt{(K_{spr})^2 (m_2 + m_1)^2 - 4 \cdot K_{pr}(m_2 + m_1) \cdot m_1 \cdot m_2}}{2 \cdot m_1 \cdot m_2}. \quad (6)$$

If the expressions of the solutions (6) of the characteristic equation (5) were real, then the solution of the differential equation (4) was:

$$u = C_1 \cdot e^{\lambda_1 \cdot t} + C_2 \cdot e^{\lambda_2 \cdot t} + \frac{m_1}{K_{pr} \cdot (m_2 + m_1)} \cdot f(t). \quad (7)$$

The constant of integrations of equation (7) were determined, taking into account the substitution (3):

$$y_1 - y_2 = C_1 \cdot e^{\lambda_1 \cdot t} + C_2 \cdot e^{\lambda_2 \cdot t} + \frac{m_1}{K_{pr} \cdot (m_2 + m_1)} \cdot f(t). \quad (8)$$

$$\dot{y}_1 - \dot{y}_2 = C_1 \cdot \lambda_1 \cdot e^{\lambda_1 \cdot t} + C_2 \cdot \lambda_2 \cdot e^{\lambda_2 \cdot t}. \quad (9)$$

The constants of integration are given as follows:

$$C_1 = \frac{f(t) \cdot m_1}{K_{pr} \cdot (m_2 + m_1)} \cdot \frac{\lambda_2}{\lambda_2 - \lambda_1}, \quad C_2 = -\frac{f(t) \cdot m_2}{K_{pr} \cdot (m_2 + m_1)} \cdot \frac{\lambda_1}{\lambda_2 - \lambda_1}. \quad (10)$$

After inverse transformation and using substitution (3), the constants of integration (10) and the solution (7) of the differential equation (4), the following was obtained:

$$y_1 - y_2 = f(t) \cdot \frac{m_1}{K_{pr} \cdot (m_1 + m_2)} \cdot \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \cdot e^{\lambda_1 \cdot t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} \cdot e^{\lambda_2 \cdot t} + 1\right),$$

$$\dot{y}_1 - \dot{y}_2 = f(t) \cdot \frac{m_1}{K_{pr} \cdot (m_1 + m_2)} \cdot \frac{\lambda_1 \cdot \lambda_2}{\lambda_2 - \lambda_1} \cdot \left(e^{\lambda_1 \cdot t} - e^{\lambda_2 \cdot t}\right). \quad (11)$$

The obtained expressions (11) were substituted into the differential equations of system (2); after transformation and grouping, the following was obtained**:**

$$\frac{d^2y_1}{dt^2} = -\frac{f(t)}{m_1 + m_2}\left(\frac{K_{spr}}{K_{pr}} \cdot \frac{\lambda_1 \cdot \lambda_2}{\lambda_2 - \lambda_1}\left(e^{\lambda_1 \cdot t} - e^{\lambda_2 \cdot t}\right) + \frac{1}{\lambda_2 - \lambda_1}\left(\lambda_2 e^{\lambda_1 \cdot t} - \lambda_1 e^{\lambda_2 \cdot t}\right) + 1\right)$$

$$\frac{d^2y_2}{dt^2} = \frac{f(t) \cdot m_1}{m_2(m_1 + m_2)}\left(\frac{K_{spr}}{K_{pr}} \cdot \frac{\lambda_1 \cdot \lambda_2}{\lambda_2 - \lambda_1}\left(e^{\lambda_1 \cdot t} - e^{\lambda_2 \cdot t}\right) + \frac{1}{\lambda_2 - \lambda_1}\left(\lambda_2 e^{\lambda_1 \cdot t} - \lambda_1 e^{\lambda_2 \cdot t}\right) + 1\right) + \frac{f(t)}{m_2}. \quad (12)$$

Each differential equation in system (12) was integrated twice. The constants of integration were defined based on the conditions: $t = 0$, $y_1 = y_2 = 0, \dot{y}_1 = \dot{y}_2 = 0$

$$y_1 = -\frac{f(t)}{m_1 + m_2}\left(\frac{K_{spr}}{K_{pr}} \cdot \frac{\lambda_1 \cdot \lambda_2}{\lambda_2 - \lambda_1}\left(\frac{e^{\lambda_1 \cdot t}}{\lambda_1^2} - \frac{e^{\lambda_2 \cdot t}}{\lambda_2^2}\right) + \frac{1}{\lambda_2 - \lambda_1}\left(\frac{\lambda_2}{\lambda_1^2} e^{\lambda_1 \cdot t} - \frac{\lambda_1}{\lambda_2^2} e^{\lambda_2 \cdot t}\right) + \frac{t^2}{2} - \frac{K_{spr}}{K_{pr}} \cdot t - \right.$$

$$\left. + -\frac{\lambda_2 + \lambda_1}{\lambda_1 \cdot \lambda_2} \cdot t - \frac{K_{spr}}{K_{pr}} \cdot \frac{\lambda_2 + \lambda_1}{\lambda_1 \cdot \lambda_2} - \frac{\lambda_2^2 + \lambda_1 \cdot \lambda_2 + \lambda_1^2}{\lambda_1^2 \cdot \lambda_2^2}\right), \quad (13)$$

$$y_2 = \frac{f(t) \cdot m_1}{m_2 \cdot (m_1 + m_2)}\left(\frac{K_{spr}}{K_{pr}} \cdot \frac{\lambda_1 \cdot \lambda_2}{\lambda_2 - \lambda_1}\left(\frac{e^{\lambda_1 \cdot t}}{\lambda_1^2} - \frac{e^{\lambda_2 \cdot t}}{\lambda_2^2}\right) + \frac{1}{\lambda_2 - \lambda_1}\left(\frac{\lambda_2}{\lambda_1^2} e^{\lambda_1 \cdot t} - \frac{\lambda_1}{\lambda_2^2} e^{\lambda_2 \cdot t}\right) + \right.$$

$$+\frac{t^2}{2}-\frac{K_{spr}}{K_{pr}}\cdot t-\frac{\lambda_2+\lambda_1}{\lambda_1\cdot\lambda_2}\cdot t-\frac{K_{spr}}{K_{pr}}\cdot\frac{\lambda_2+\lambda_1}{\lambda_1\cdot\lambda_2}-\frac{\lambda_2^2+\lambda_1\cdot\lambda_2+\lambda_1^2}{\lambda_1^2\cdot\lambda_2^2}\Big). \quad (14)$$

The obtained dependences (13) and (14) characterise the movement of masses m_1 and m_2 under the applied impulse $f(t)$ provided that the solutions of the characteristic equation (5) were real numbers. The case when the solutions of the characteristic equation (5) were complex numbers was also considered. In that case, the roots were as follows:

$$\lambda_{1,2}=\alpha\pm\beta\cdot i. \quad (15)$$

The homogeneous system of equations (2) was solve analytically in the form of $y_1 = A\cdot e^{\lambda t}$, $y_2 = B\cdot e^{\lambda t}$, for both the complex roots from equation (15) and the real roots from equation (6). After substituting into the first equation of system (2), the following was obtained:

$$e^{\lambda\cdot t}\cdot\left(A\cdot(\lambda^2+n_1\cdot\lambda+K_1^2)-B\cdot(n_1\cdot\lambda+K_1^2)\right)=0. \quad (16)$$

From equation (16), the following was taken into account:

$$B=A\cdot(\lambda^2+n_1\cdot\lambda+K_1^2)/(n_1\cdot\lambda+K_1^2). \quad (17)$$

Considering the values of the roots of the characteristic equation, the general solution to the homogeneous system of differential equations (2) was expressed as follows:

$$y_{10}=A_0+A_1\cdot e^{\lambda_1\cdot t}+A_2\cdot e^{\lambda_2\cdot t},$$
$$y_{20}=y_{10}+A_1\cdot\frac{\lambda_1^2}{n_1\cdot\lambda_1+K_1^2}\cdot e^{\lambda_1\cdot t}+A_2\cdot\frac{\lambda_2^2}{n_1\cdot\lambda_2+K_1^2}\cdot e^{\lambda_2\cdot t}. \quad (18)$$

A partial solution satisfying the initial conditions: $t = 0$, $y_{10}(0)=0, \dot{y}_{10}(0)=0, y_{20}(0)=0, \dot{y}_{20}(0)=1$ was found. Based on expressions (18), a system of algebraic equations was formed:

$$\begin{cases} A_0+A_1+A_2=0 \\ \lambda_1\cdot A_1+\lambda_2\cdot A_2=0 \\ \frac{\lambda_1^2}{n_1\cdot\lambda_1+K_1^2}\cdot A_1+\frac{\lambda_2^2}{n_1\cdot\lambda_2+K_1^2}\cdot A_2=0 \\ \frac{\lambda_1^3}{n_1\cdot\lambda_1+K_1^2}\cdot A_1+\frac{\lambda_2^3}{n_1\cdot\lambda_2+K_1^2}\cdot A_2=1 \end{cases}. \quad (19)$$

The coefficients in equations (18) were determined from the system of equations (19). Matrices were formed to calculate the determinants:

$$\Delta=\begin{pmatrix}\lambda_1 & \lambda_2\\ \frac{\lambda_1^3}{n_1\cdot\lambda_1+K_1^2} & \frac{\lambda_2^3}{n_1\cdot\lambda_2+K_1^2}\end{pmatrix},\quad \Delta 1=\begin{pmatrix}0 & \lambda_2\\ 1 & \frac{\lambda_2^3}{n_1\cdot\lambda_2+K_1^2}\end{pmatrix},\quad \Delta 2=\begin{pmatrix}\lambda_1 & 0\\ \frac{\lambda_1^3}{n_1\cdot\lambda_1+K_1^2} & 1\end{pmatrix}. \quad (20)$$

Then the coefficients of equations (18) were determined**:**

$$A_0=-\frac{\Delta 1+\Delta 2}{\Delta}, A_1=\frac{\Delta 1}{\Delta}, A_2=\frac{\Delta 2}{\Delta}. \quad (21)$$

The solution of the set of equations (2) was as follows:

$$y_1(t)=C_0+C_1\cdot e^{\lambda_1\cdot t}+C_2\cdot e^{\lambda_2\cdot t}+h_2\cdot\int_0^t y_{10}(t-z)\cdot f(z)\cdot dz$$
$$y_2(t)=C_0+C_1\cdot\left(1+\frac{\lambda_1^2}{n_1\cdot\lambda_1+K_1^2}\right)\cdot e^{\lambda_1\cdot t}+C_2\cdot\left(1+\frac{\lambda_2^2}{n_1\cdot\lambda_2+K_1^2}\right)\cdot e^{\lambda_2\cdot t}+$$
$$+h_2\cdot\int_0^t y_{20}(t-z)\cdot f(z)\cdot dz. \quad (22)$$

)

The constants of the solutions (22), C_0, C_1, and C_2, were determined from the initial conditions. Given that at $t = 0$, $y_1(0)=\dot{y}_1(0)=y_2(0)=\dot{y}_2(0)=0$, the constants C_0, C_1, and C_2 were also equal to zero. The results of the integration of equations (21) constituted their solution:

$$y_1(z)=h_2\cdot\int_0^t y_{10}(t-z)\cdot f(z)\cdot dz=h_2\cdot\left(A_0\cdot z-\frac{A_1}{\lambda_1}\cdot e^{\lambda_1\cdot(t-z)}-\frac{A_2}{\lambda_2}\cdot e^{\lambda_2\cdot(t-z)}\right)$$
$$y_2(z)=y_1(z)-h_2\cdot\left(A_1\cdot\frac{\lambda_1}{n_1\cdot\lambda_1+K_1^2}\cdot e^{\lambda_1\cdot(t-z)}+A_2\cdot\frac{\lambda_2}{n_1\cdot\lambda_2+K_1^2}\cdot e^{\lambda_2\cdot(t-z)}\right). \quad (23)$$

It was supposed that the nature of the oscillation of the applied force corresponded to the dependence shown in Fig. 2, which was analytically described by equation (24):

$$f(t)=\begin{cases}1, n\cdot T<t<n\cdot T+\tau\\ 0, n\cdot T+\tau<t<(n+1)\cdot T\end{cases}. \quad (24)$$

where $n = [\tau/T]$ – the integer component of a number τ/T.

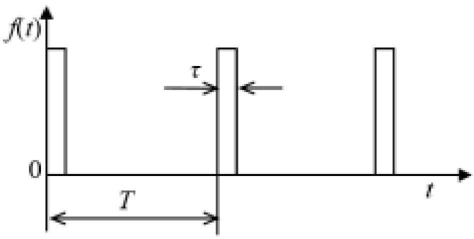

Fig. 2. The nature of the f(t) impulse of the applied force: *τ* – duration of the force impulse; *T* - impulse period
Source: developed by the authors based on Dmytriv *et al.* [8]

For the oscillation character of the applied force corresponding to the discrete function (24), the solution (23) was formulated with consideration of the constraints imposed by the analytical form of the discrete function (24) representing the applied force. Additionally, in equations (22), the variable *z* represented the time interval from 0 to the given moment, while *t* in equations (23) characterised the duration of the force acting on the element with mass m_1; in this case, $t = \tau$. Accordingly, the equations describing the motion of the masses were expressed as the system (25) and (26), where z was replaced by *t*:

$$y_1(t) = \begin{cases} y_1(z) - y_1(nT) + \sum_0^n \left(y_1(nT+\tau) - y_1(nT)\right), nT \le t \le nT+\tau \\ \sum_0^n \left(y_1(nT+\tau) - y_1(nT)\right), nT+\tau \le t \le (n+1)T \end{cases}, \quad (25)$$

$$y_2(t) = \begin{cases} y_2(z) - y_2(nT) + \sum_0^n \left(y_2(nT+\tau) - y_2(nT)\right), nT \le t \le nT+\tau \\ \sum_0^n \left(y_2(nT+\tau) - y_2(nT)\right), nT+\tau \le t \le (n+1)T \end{cases} \quad (26)$$

The obtained system of differential equations forms the basis for further numerical simulation of the oscillatory process under varying impulse excitation conditions. This mathematical model enables the analysis of the influence of system parameters – such as mass, stiffness, damping, and impulse duration – on the dynamic behaviour of the two-mass system. The model is applied to investigate the system's response under different initial conditions and parameter combinations.

Results and Discussion

The article details the research results related to the Kelvin-Voigt system. The movement of masses m_1 and m_2 was simulated using the provided data: masses: $m_1 = 4000$ N and $m_2 = 1000$ N; coefficient of damper resistance $K_{spr} = 3000$ N·s/m; elasticity coefficient $K_{pr} = 50$ kN/m. The square of the frequency of free oscillations, respectively, for masses of m_1 and m_2 was $K_1^2 = 12.5\ s^{-2}$ and $K_2^2 = 50\ s^{-2}$, coefficients of damping oscillations, respectively $n_1 = 0.75\ s^{-1}$ and $n_2 = 3\ s^{-1}$. The solution's roots and the coefficients of the equation are presented as follows: $\lambda_1 = -1.875 - 7.68 \cdot i$; $\lambda_2 = -1.875 + 7.68 \cdot i$; $A_0 = -0.03$; $A_1 = 0.015 + 3.668 \cdot i \cdot 10^{-3}$; $A_2 = 0.015 - 3.668 \cdot i \cdot 10^{-3}$. With consideration of the coefficients and roots of the $z = t$ and $t = \tau$ (Fig. 2), values the (23) equations will have a following view:

$$\begin{aligned} y_1(t) &= 1\left(-0.03t - \frac{(0.015+3.668i\cdot 10^{-3})}{(-1.875-7.68i)} \cdot e^{(-1.875-7.68i)\cdot(\tau-t)} - \right. \\ &\left. - \frac{(0.015-3.668i\cdot 10^{-3})}{(-1.875+7.68i)} \cdot e^{(-1.875+7.68i)\cdot(\tau-t)}\right) \\ y_2(t) &= y_1(t) - 1\left((0.015+3.668i\cdot 10^{-3})\frac{(-1.875-7.68i)}{0.75(-1.875-7.68i)+12.5} \times \right. \\ &\left. \times e^{(-1.875-7.68i)\cdot(\tau-t)} + (0.015-3.668i\cdot 10^{-3}) \times \frac{(-1.875+7.68i)}{0.75(-1.875+7.68i)+12.5} \cdot e^{(-1.875+7.68i)\cdot(\tau-t)}\right) \end{aligned} \quad (27)$$

The simulation results of the oscillatory behaviour of the Kelvin-Voigt body mass based on the previously mentioned parameters with a duration of $\tau = 0.1$ s of the pulse of the applied force are shown in Fig. 3a. Similarly, mass oscillations were modelled using alternative system parameters, and the results are presented in Fig. 3b, 3c, 3d.

The analysis shows that the obtained model (25), (26) of the Kelvin-Voigt mass oscillation enables modelling in the entire range of real and complex roots of the equations. The simulation results showed that with a high coefficient of damper resistance $K_{spr} \ge 3000$ N·s/m and a low coefficient of elasticity $K_{pr} \le 50$ kN/m and significant masses of m_1 and m_2, the oscillations are continuous (Fig. 3a). Under the condition of increasing the coefficient of elasticity and the resistance coefficient of the damper, oscillation amplitude decreases as the system exhibits damping (Fig. 3b, 3c, 3d). An increase in the mass of the oscillating bodies increases their displacement of y_1 and y_2. Accordingly, the oscillation damping time increases. Analysis of the study results indicated that the oscillation amplitude decreases as the square of the natural oscillation frequency of the mass diminishes. An increase in the elasticity coefficient of the elastic element results in a higher natural oscillation frequency of the masses, thereby enhancing their motion. An increase in masses leads to a decrease in their movements, the repeatability of oscillations increases. The repeatability of the oscillations of bodies increases with the increase in the duration of the pulse of the mass disturbance force. To model the oscillation of the Kelvin-Voigt body masses with real values of the roots (5) of the characteristic equation (6), solutions (14) and (15) can be taken.

The Kelvin-Voigt model remains a foundational tool for simulating the dynamic behaviour of systems with elastic and viscous characteristics. Despite its analytical simplicity for idealised single-degree-of-freedom systems, the extension to multi-mass or non-linear configurations, as in the present study, significantly complicates the analytical resolution.

This has led to diverse methodological approaches in the literature, combining both analytical and numerical strategies, which merit detailed comparison. In the work by Jugulkar *et al.*, a suspension system with variable stiffness and damping was analysed for automotive use [9]. Their model, while physically implementable, required extensive numerical parametrisation due to the complexity of time-dependent boundary conditions. Similarly, Madeira & Coda used Kelvin-type visco-elasticity coupled with Lagrange multipliers to control nonlinear vibrations [10]. Unlike the present study, they focused on control mechanisms, while our research emphasises free oscillation dynamics under impulse loading.

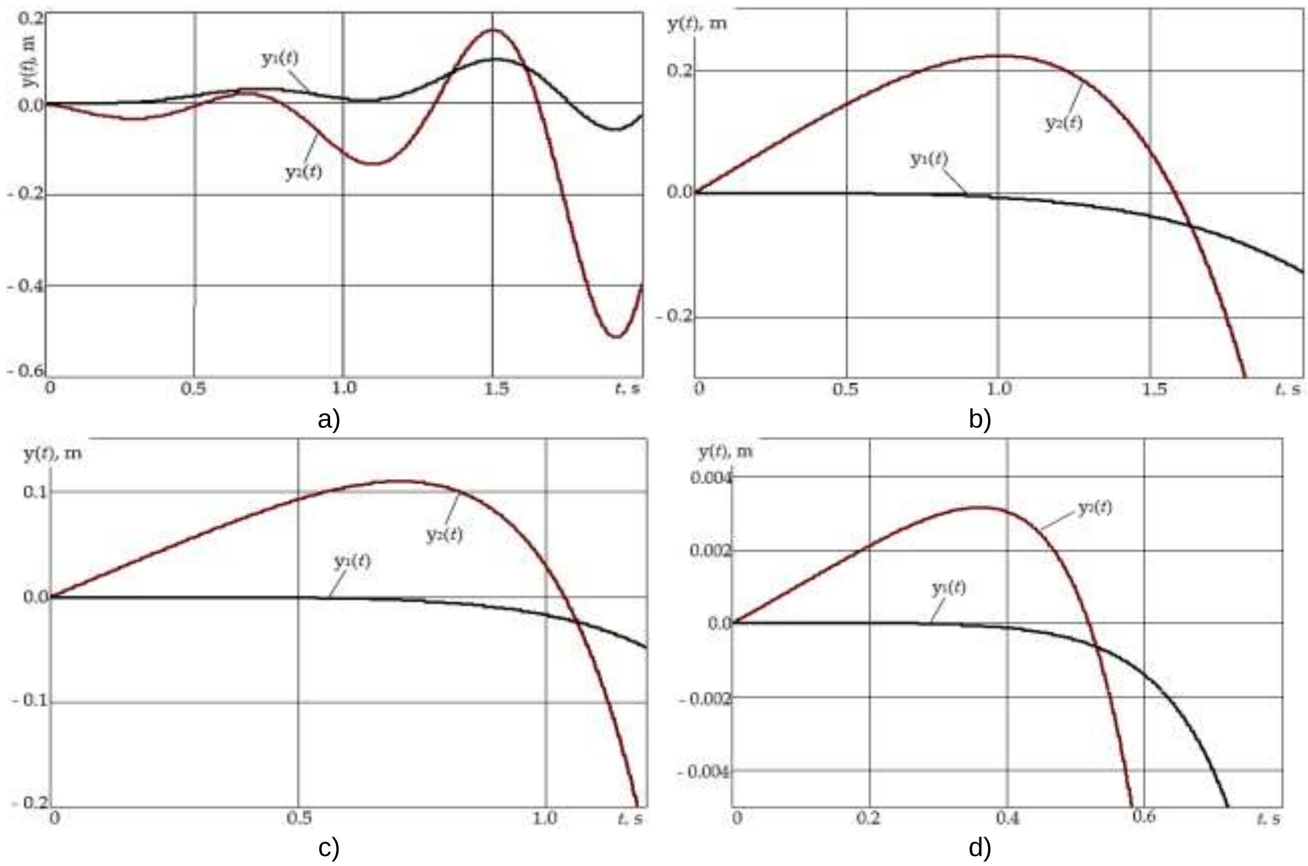

Fig. 3. Results of simulation of the Kelvin-Voigt body mass oscillation based on the system parameters

Note: a) m_1 = 4000 N, m_2 = 1000 N, K_{spr} = 3000 N·s/m, K_{pr} = 50 kN/m, $K_1^2 = 12.5\ s^{-2}$, $K_2^2 = 50\ s^{-2}$, n_1 = 0.75 s^{-1} and n_2 = 3 s^{-1}, λ_1 = - 1.875 – 7.68·i, λ_2 = - 1.875 + 7.68·i, A_0 = -0.03, A_1 = 0.015 + 3.668·i·10^{-3}, A_2 = 0.015 - 3.668·i·10^{-3};

b) m_1 = 4000 N, m_2 = 400 N, K_{spr} = 800 N·s/m, K_{pr} = 500 N/m, $K_1^2 = 0.125\ s^{-2}$, $K_2^2 = 1.25\ s^{-2}$, n_1 = 0.2 s^{-1}, n_2 = 2 s^{-1}, λ_1 = - 1.1 – 0.406·i, λ_2 = - 1.1 + 0.406·i, A_0 = 0.027; A_1 = -0.014 - 0.037·i, A_2 = - 0.014 + 0.037·i;

c) m_1 = 4000 N, m_2 = 410 N, K_{spr} = 1800 N·s/m, K_{pr} = 700 N/m, $K_1^2 = 0.175\ s^{-2}$, $K_2^2 = 1.707\ s^{-2}$, n_1 = 0.45 s^{-1}, n_2 = 4.39 s^{-1}, λ_1 = - 4.414, λ_2 = - 0.426, A_0 = 1.839·10^{-3}; A_1 = 1.967·10^{-4}, A_2 = -2.045·10^{-3};

d) m_1 = 4000 N, m_2 = 410 N, K_{spr} = 4000 N·s/m, K_{pr} = 4030 N/m, $K_1^2 = 1.008\ s^{-2}$, $K_2^2 = 9.829\ s^{-2}$, n_1 = 1 s^{-1}, n_2 = 9.756 s^{-1}, λ_1 = - 9.631, λ_2 = - 1.125, A_0 = 9.963·10^{-4}; A_1 = 1.318·10^{-4}, A_2 = -1.128·10^{-3}

Source: developed by the authors

Monsia & Kpomahou addressed nonlinear damping in mechanical systems using polynomial approximations [11]. Their approach differs fundamentally by relying entirely on numerical simulations, whereas our method integrates an exact analytical solution for both real and complex roots. Herisanu *et al.* offered an analytical solution for harmonic excitation in a one-mass Kelvin-Voigt model [12]. In contrast, our study introduces impulse-type forces of variable frequency and duration, which are rarely addressed in closed-form. Gomez-Aguilar *et al.* applied Laplace transforms in a fractional-order mass-spring-damper model to describe forced vibrations [13]. Although useful in frequency domain analysis, this technique loses resolution in transient impulse loading scenarios, which are central to our approach. Hamedi *et al.* proposed a reduced-order Newton iteration scheme in Simulink to optimise damper design [14]. Unlike their purely numerical strategy, we preserve analytical traceability in model construction and solution.

From the perspective of fractional visco-elasticity, several authors have extended the classical Kelvin-Voigt model by incorporating non-integer derivatives to capture memory effects and nonlocal behaviour of materials. Stankovic & Atanackovic analysed the dynamics of a visco-elastic rod described by a generalised constitutive equation that reflects hereditary material properties [15]. Their study highlighted how fractional calculus enables modelling of long-term stress relaxation and creep phenomena, which are inadequately described by conventional integer-order models. Similarly, Rossikhin & Shitikova investigated damped vibrations in thin elastic structures embedded in a fractional derivative visco-elastic medium, demonstrating the applicability of such models to layered and composite systems [16]. Both studies contribute valuable rheological frameworks and advance theoretical understanding of fractional-order dynamics. However, they are predominantly focused on material-level behaviour rather than mass-interaction systems, and often rely on semi-analytical or purely numerical approaches due to the absence of tractable closed-form solutions. In contrast, the present study offers an exact analytical formulation for a two-mass Kelvin–Voigt system subject to discrete impulse excitation, providing a rare instance where both fractional-like temporal behaviour and structural dynamics are captured without recourse to full numerical discretisation, which confirms the effectiveness of the analytical approach [17]. This comparison underscores the utility of the proposed model as a middle ground between purely empirical damping descriptions and abstract fractional formulations. In contrast to the above, current study presents a closed-form analytical solution for a two-mass system with impulse excitation, including real and complex eigenvalues. This makes it suitable for benchmarking visco-elastic models where control of both oscillation amplitude and duration is required. Furthermore, Bobyleva & Shamaev discussed integral memory effects in damping models, which are analogous to our discrete pulse modelling in that both represent temporally extended system responses [18].

A more applied context is seen in Wu *et al.* [4], who experimentally validated damping in turbine blades using under-platform dampers. While their study focuses on structural integrity, our model targets configurational optimisation in mechanical system design. Similarly, Zhang *et al.* [19] studied hydropneumatic suspensions with damping noncoincidence, introducing system-specific damping factors that can also be interpreted through our variable parameter analysis. Finally, Dmytriv *et al.* proposed spring-damper models in agricultural machines, validating them experimentally [8]. Their results support the use of simplified visco-elastic analogues for practical control scenarios, a notion reinforced by our analytical formulations. In summary, while other studies rely heavily on numerical simulation or fractional calculus to address visco-elastic complexity, the presented work bridges this gap by offering a rare analytical solution under discrete impulsive loads with varying system parameters.

The two-mass elastic-damper system was solved using the Euler method for the roots of the characteristic equation with complex numbers. This method has no universal solution and no equivalents. Unlike previous studies, where the applied perturbation force was described in the form of harmonic oscillations, here it was proposed a solution for the perturbation force in the form of discrete single pulses of different durations and different arrival frequencies from one pulse to *n* pulses.

The comparative analysis confirms that despite the wide diversity of modelling approaches – from purely numerical schemes to fractional visco-elastic formulations – few studies provide exact analytical solutions for systems with discrete impulse excitation. The present research fills this methodological gap by offering a mathematically rigorous and physically interpretable model applicable across a wide range of system configurations. This solution structure ensures both precision and adaptability in the simulation of dynamic responses under variable damping and stiffness parameters.

Conclusions. The article presents an analytical solution to the system of differential equations describing the coupled oscillations of masses interconnected by parallel elastic and viscous elements. Such a configuration corresponds to the classical Kelvin-Voigt model. The analytical model enables the simulation of technical systems operating on this principle, such as suspensions in automobiles and other vehicles designed for various applications. The mathematically derived analytical model facilitates the optimisation of suspension system designs. Compared to alternative mathematical approaches, the analytical solution of the system of differential equations describing the motion of the system's masses allows for comprehensive analysis across a broad spectrum of structural parameter values.

In prior studies, the applied disturbance force was characterised by harmonic oscillations. In contrast, we propose a solution that models the disturbance input as discrete single pulses of varying durations and repetition frequencies, ranging from a single pulse to multiple pulses (up to n).

The movement of masses m_1 and m_2 was simulated based on the given data: masses: $m_1 = 4000$ N and $m_2 = 1000$ N; coefficient of damper resistance $K_{spr} = 3000$ N·s/m; elasticity coefficient $K_{pr} = 50$ kN/m. The square of the frequency of free oscillations, respectively, for masses of m_1 and m_2 was $K_1^2 = 12.5\ s^{-2}$ and $K_2^2 = 50\ s^{-2}$, coefficients of damping oscillations, respectively $n_1 = 0.75\ s^{-1}$ and $n_2 = 3\ s^{-1}$. The roots of the solution and coefficients of the equation are as follows: $\lambda_1 = -1.875 - 7.68 \cdot i$; $\lambda_2 = -1.875 + 7.68 \cdot i$; $A_0 = -0.03$; $A_1 = 0.015 + 3.668 \cdot i \cdot 10^{-3}$; $A_2 = 0.015 - 3.668 \cdot i \cdot 10^{-3}$. An increase in the coefficient of elasticity of the elastic element leads to an increase in the frequency of natural oscillations of the masses, which increases the displacement of these masses. An increase in masses leads to a decrease in their displacements, the repeatability of oscillations increases. The repeatability of the oscillations of bodies increases with the increase in the duration of the pulse of the mass disturbance force.

Further studies of this problem should be focused on a wider configuration of the combination of different bodies and their different number, which will make it possible to optimise the designs of technical systems that use oscillations, vibrations, damping, and creep. This will allow interested person to manage these processes accordingly.

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МАТЕМАТИЧНА МОДЕЛЬ ПРУЖНО-ДЕМПФЕРНОЇ СИСТЕМИ НА ПРИКЛАДІ ТІЛА КЕЛЬВІНА-ФОЙГТА

Механічні системи, які поєднують конструктивні елементи з пружними та в'язкими характеристиками та працюють в умовах динамічних та імпульсних навантажень, потребують точних математичних моделей для прогнозування їх коливальної поведінки. Метою дослідження було розроблення математичної моделі пружно-демпферної системи на основі тіла Кельвіна-Фойгта для опису в'язкопружної поведінки матеріалу та аналізу впливу параметрів жорсткості й демпфування на динамічну відповідь системи. У роботі використано аналітичні методи розв'язування диференціальних рівнянь, аналіз характеристичних рівнянь для дійсних і комплексних коренів, а також алгебраїчні перетворення для побудови загального та частинного розв'язків. В результаті досліджень отримано аналітичний розв'язок задачі визначення динамічної реакції мас при імпульсному навантаженні. Запропонована модель враховувала змінні параметри жорсткості, демпфування та мас, а також розрізняє режими з дійсними та комплексними власними значеннями. Встановлено, що підвищення коефіцієнта демпфування зменшує амплітуду та прискорює згасання коливань, тоді як збільшення маси продовжує час згасання. Досліджено вплив тривалості імпульсу та частоти його повторення на форму коливального процесу. Результати моделювання підтверджують точність аналітичних залежностей і дозволяють змоделювати перехідні процеси за різних умов. У порівнянні з числовими підходами, запропонований метод надав ширші можливості для аналізу та контролю систем. Розроблена модель може бути застосована при дослідженні технічних систем з довільною конфігурацією з'єднання пружно-в'язких тіл та їх кількістю, що дозволить оптимізувати проектування технічних систем, що використовують коливання, вібрації, демпфування.

Ключові слова: *двомасова система, диференціальні рівняння, характеристичне рівняння, коефіцієнт пружності, демпфер.*

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